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Anisotropic thermoconvective effects on the onset of double diffusive convection in a porous medium

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Abstract—The linear stability of the thermodiffusive equilibrium of a binary mixture of two miscible fluids in a horizontal plane porous layer is investigated. The linear theory is based on the normal mode analysis under the small amplitude assumption. The effect of anisotropic thermo-convective currents on the stability is obtained. The effect of Prandtl number, ratio of diffusivities and separation parameter are presented graphically. It is found that the thermo-convective currents have a stabilizing effect as well as a destabilizing effect with respect to the case in which these currents are absent. We also find that either a stationary instability or oscillatory instability can occur as the first bifurcation. Some of the results about the former problems are deduced.

1. INTRODUCTION

IF GRADIENTS of two stratifying agencies, such as heat and salt, having different diffusivities are simultaneously present in a fluid layer, a variety of interesting convective phenomena can occur which are not possible in single component fluids. The case of two stratifying agencies has been the subject of extensive studies which have been reviewed by Turner [1–3]. During the last decade, there has been considerable interest in systems showing an oscillatory instability as the first bifurcation in various branches of fluid mechanics and condensed matter physics such as convective instabilities in mixtures of two fluids, in superfluid mixtures and in various types of liquid crystals like uniaxial nematics and cholesteries and smectics. The quality of the single crystal produced from melts is limited by chemical and structural inhomogeneities. The defect generation depends on heat and mass transfer rates during solidification. These fluxes are mainly governed by convective phenomena of the liquid phase during processing.

In the case of binary mixtures thermo-convective equilibrium is possible due to the presence of gradients of heat and also of concentration. The possibility of maintaining the stability, or generating instability, is more varied and pronounced as diffusive thermo-convective currents can be set up due to the critical differences in temperature or concentration or both, imposed on the boundaries of the layer. Further, these critical differences can be affected by anisotropic effects within the mixture itself, due to interactions between the thermal and concentration gradients known as Soret and Dufour effects. McDougall [4], Brand and Steinberg [5] and Rudraiah and Malashetty [6] have studied the effect of these cross-diffusions on double-diffusive convection in viscous fluids and porous layers. But these authors have ignored the effect of anisotropy. In this present paper we investigate the effect of anisotropic thermo-convective currents on thermodiffusive equilibrium in a horizontal porous layer using linear stability analysis.

2. MATHEMATICAL FORMULATION

The basic equations describing the dynamics of a binary fluid mixture in a horizontal porous layer bounded between two free boundaries at $z = 0$ and d taking into account the effects of interaction between thermal convection and thermoconvective diffusivity in the Boussinesq approximation are given by (Brand and Steinberg [5], Maiellaro and Palese [7]):

$$\rho_0\left[\left(\frac{\partial \mathbf{q}}{\partial t}\right)+(b/\sqrt{k})\mathbf{q}|\mathbf{q}|\right] = -\nabla p+\rho\mathbf{g}-(\mu/k)\mathbf{q} \tag{1}$$

$$(\partial T/\partial t)+\mathbf{q}\cdot\nabla T = (K_T+N\lambda^2K_c)\nabla^2T+N\lambda K_c\nabla^2C \tag{2}$$

$$(\partial C/\partial t)+\mathbf{q}\cdot\nabla C = K_c\nabla^2T+K_c\nabla^2C \tag{3}$$

$$\nabla\cdot\mathbf{q} = 0 \tag{4}$$

$$\rho = \rho_0[1-\alpha_T(T-T_0)+s\alpha_c(C-C_0)] \tag{5}$$

with $s = \pm 1$ according to which density of the solute is greater or smaller than the solvent. We can add to equations (1)–(5) the following boundary conditions:

$$w = \partial^2 w/\partial z^2 = 0 \quad \text{at } z = 0 \text{ and } d$$
$$T = T_0,\quad C = C_0 \quad \text{at } z = 0$$
$$T = T_d,\quad C = C_d \quad \text{at } z = d. \tag{6}$$

Equations (1)–(5) allow the stationary solutions given by

NOMENCLATURE

a	dimensionless wave number, $(k_1^2+k_2^2)^{1/2}$
b	drag co-efficient
C	concentration field
d	layer thickness
F	porous parameter, d^2/k
g	acceleration due to gravity
K_T	thermometric conductivity
K_c	convection co-efficient
k	permeability of a porous medium
k_1, k_2	wave numbers in horizontal plane
Le	Lewis numbers, Pr/Sc
P	pressure
Pr	Prandtl number, ν/K_T
$\mathbf{q}$	velocity components, (u, v, w)
R	thermal Rayleigh number, $\alpha_\tau g\beta d^4/\nu K_T$
R_L	Lapwood Rayleigh number, R/F
R_c	solute Rayleigh number, $\alpha_c g\Gamma d^4/\nu K_c$
R_{cL}	R_c/F
S	dimensionless perturbed concentration
s	± 1
Sc	Schmidt number, ν/K_c
T	temperature field
x, y, z	coordinates
y_n	$n^2\pi^2+a^2$.

Greek symbols

α_τ	co-efficient of thermal expansion
α_c	solute analog of α_T
β	temperature gradient
Γ	concentration gradient
θ	dimensionless temperature perturbation
λ	thermoconvection co-efficient
ν	kinematic viscosity
ρ	density
σ	frequency, pd^2/ν
ψ_1	separation parameter, α_c/α_T
ψ_2	Dufour parameter, $N\lambda^2$
ω	frequency.

Other symbols

∇^2	$\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$
∇_1^2	$\nabla^2-\partial^2/\partial z^2$.

Subscripts

0	indicates reference values
d	indicates values at $z = d$
$'$	perturbed variables.

$$\mathbf{q} = 0, \quad T = T_0-\beta z, \quad C = C_0-\Gamma z,$$
$$\nabla p = \rho_0 g[1+(\alpha_T\beta-s\alpha_c\Gamma)z] \tag{7}$$

where $\beta = (T_0-T_d)/d$, $\Gamma = (C_0-C_d)/d$.

Now we shall study the linear stability of stationary solution (7) and examine the effect of anisotropic thermoconvective currents on this stability. For the deviations from the stationary state we find the following linearized equations:

$$(\partial\mathbf{q}/\partial t) = (1/\rho_0)\nabla p'+g[-\alpha_T T'+s\alpha_c C']-(\nu/k)\mathbf{q} \tag{8}$$
$$(\partial T'/\partial t) = (K_T+N\lambda^2 K_c)\nabla^2 T'+N\lambda K_c\nabla^2 C'+\beta w \tag{9}$$
$$(\partial C'/\partial t) = K_c\nabla^2 T'+K_c\nabla^2 C'+\Gamma w \tag{10}$$
$$\nabla\cdot\mathbf{q} = 0. \tag{11}$$

Eliminating the pressure from (8) it follows that

$$(\partial\nabla^2 w/\partial t) = -(\nu/k)\nabla^2 w+\alpha_T q\nabla_1^2 T'-\mathbf{q}\alpha_c s\nabla_1^2 C'. \tag{12}$$

From (6) we obtain the following boundary conditions for the perturbations:

$$w = d^2w/dz^2 = T' = C' = 0 \quad \text{at } z = 0, d. \tag{13}$$

3. LINEAR STABILITY ANALYSIS

For the perturbation equation (9), (10) and (12) we look for solutions of the form:

$$(w, T', C') = [(\nu/d)w(z), \quad (\nu d/K_T)\theta(z), \quad (\nu d/K_c)S(z)]\exp[i(k_1x+k_2y)+pt] \tag{14}$$

and obtain the following equations:

$$[(D^2-a^2)(1+N\lambda^2\,Pr/Sc)-\sigma Pr]\theta = -\beta w-N\lambda(D^2-a^2)S \tag{15}$$
$$(D^2-a^2-\sigma Sc)S = -\Gamma w-\lambda(Pr/Sc)(D^2-a^2)\theta \tag{16}$$
$$(\sigma+F)(D^2-a^2)w = -(\alpha_T g a^2 d^6/\nu K_T)\theta + (\alpha_c g s a^2 d^6/\nu K_c)S. \tag{17}$$

The boundary conditions (13) become:

$$w = D^2 w = \theta = S = 0 \quad \text{at } z = 0, 1. \tag{18}$$

Finally, eliminating θ and S between equations (15)–(17) we obtain a single differential equation for the stability problem as:

$$\begin{aligned}(\sigma+F)(D^2-a^2)\{(D^2-a^2-\sigma Sc)[(D^2-a^2)\\ \times(1+\lambda^2 N\,Pr/Sc)-\sigma Pr]-(D^2-a^2)N\lambda^2 Pr/Sc\}w\\ = Ra^2(D^2-a^2-\sigma Sc)w-sR_c a^2[(D^2-a^2)\\ \times(1+N\lambda^2 Pr/Sc)-\sigma Pr]w\\ +(\lambda g a^2 d^6/\nu K_T)(D^2-a^2)(-N\alpha_T\Gamma+s\beta\alpha_c)w.\end{aligned} \tag{19}$$

For wave-like perturbations, A sin $(n\pi z)$, w, and D^2w and all even derivatives becomes zero and from (19) we obtain

$$(\sigma+F)y_n\{(y_n+\sigma Sc)[y_n(1+\lambda^2 N\,Le)+\sigma Pr\}$$
$$-y_n^2 N\lambda^2\,Le = Ra^2(y_n+\sigma Sc)$$
$$-sR_c a^2[y_n(1+\lambda^2 N\,Le)+\sigma Pr]$$
$$+\lambda a^2 y_n[-NR_c\,Le\,\alpha_T/\alpha_c+R_c\,s\,\alpha_c/\alpha_T] \quad (20)$$

with $y_n = n^2\pi^2+a^2$ and $Le = Pr/Sc$.

In the critical region of neutral stability, the real part of σ must be zero. Therefore taking $\sigma = i\omega$, ω real, we examine two possibilities, i.e. $\omega = 0$ and $\omega \neq 0$, according to whether the instability occurs as a stationary convection or as an oscillatory mode. For $\omega = 0$ we obtain from (20)

$$R_L(1+\lambda s\alpha_c/\alpha_T)-R_{cL}[N\,Le\,\alpha_T/\alpha_c+s(1+\lambda^2 N\,Le)] = y_n^2/a^2. \quad (21)$$

The minimum value of R_L occurs at $a_c = \pi$ for $n = 1$ and is given by

$$R_L(1+\lambda s\alpha_c/\alpha_T)-R_{cL}[N\,Le\,\alpha_T/\alpha_c+s(1+\lambda^2 N\,Le)] = 4\pi^2. \quad (22)$$

For a single component fluid we obtain from (22) the minimum Rayleigh number

$$R_L = 4\pi^2 \quad (23)$$

given by Lapwood [8] while in the absence of cross-diffusion, we obtain the relation

$$R_L-sR_{cL} = 4\pi^2 \quad (24)$$

which was obtained by Rudraiah *et al.* [9]. For $N = 0$, i.e. in the absence of the Dufour effect, equation (22), gives

$$R_L(1+\lambda s\alpha_c/\alpha_T)-sR_{cL} = 4\pi^2. \quad (25)$$

If $\omega \neq 0$, we obtain from (20) the following relations:

$$Ra^2(1+\lambda s\alpha_c/\alpha_T)-R_c a^2[N\lambda\,Le\alpha_T/\alpha_c+s(1+N\lambda^2\,Le)+\omega^2[y_n\,Pr+Pr\,Sc\,F+Sc(1+\lambda^2 N\,Le)y_n]-Fy_n^2 = 0 \quad (26)$$

$$Ra^2\,Sc-R_c\,a^2\,s\,Pr+\omega^2 Pr\,Sc\,y_n-[Pr\,F+Sc(1+\lambda^2 N\,Le)F+y_n]y_n^2 = 0. \quad (27)$$

From these equations it follows that

$$X/Y = y_n^2/a^2 \quad (28)$$

with

$$X = R\,Sc\{Sc[Pr\,F+(1+\lambda^2 N\,Le)y_n]-\lambda s\,Pr\,y_n\alpha_c/\alpha_T\}+R_c\,Pr^2[N\lambda y_n\alpha_T/\alpha_c-s(y_n+Sc\,F)]$$
$$Y = [Pr+Sc(1+\lambda^2 N\,Le)]\{y_n^2+[Pr(y_n+Sc\,F)+Sc(1+\lambda^2 N\,Le)y_n]F\}$$
$$\omega^2 = \{-Ra^2\,Sc+R_c a^2 S\,Pr+[Pr\,F+Sc(1+\lambda^2 N\,Le)F+y_n]y_n^2\}/Pr\,Sc\,y_n. \quad (29)$$

Minimizing (28) with respect to a^2 can most conveniently be solved numerically. However, equation (28) can be simplified considerably if we assume that F is large. This condition can be satisfied for a densely packed porous medium in which case the value of F is generally greater than 10^5 (Joseph [10]). Then for oscillatory instability, equation (28) gives

$$[R_L\,Sc-s\,Pr\,R_{cL}]/[Pr+Sc(1+\lambda^2 N\,Le)] = 4\pi^2 \quad (30)$$

with $a_c^2 = \pi^2$. Equations (22) and (30) are evaluated numerically and the results are discussed in Section 4.

From (26) and (27) it follows that

$$Ra^2(1+\lambda s\alpha_c/\alpha_T)-R_c a^2[N\lambda\,Le\,\alpha_T/\alpha_c+s(1+N\lambda^2\,Le)]-Fy_n^2 = -\omega^2[Pr\,y_n+Pr\,Sc\,F+Sc(1+\lambda^2 N\,Le)y_n] \quad (31)$$

$$Ra^2\,Sc-R_c a^2\,sPr-[Pr\,F+Sc(1+N\lambda^2\,Le)F+y_n]y_n^2 = -\omega^2 Pr\,Sc\,y_n. \quad (32)$$

The right hand sides of (31) and (32) are certainly negative and this implies the negativity of the left hand sides; therefore, from these equations we have

$$R_L(1+\lambda s\,\alpha_c/\alpha_T)-R_{cL}[N\lambda\,Le\,\alpha_T/\alpha_c+s(1+\lambda^2 N\,Le)] < 4\pi^2 \quad (33)$$

$$[R_L\,Sc-sR_{cL}\,Pr]/[Pr+Sc(1+N\lambda^2\,Le)] < 4\pi^2. \quad (34)$$

We observe from (22) and (33) that conditions (33) and (34) assure, in any case ($\omega = 0$, $\omega \neq 0$), the stability.

If $\omega = 0$, in view of (33) and (24) it follows that the anisotropic thermo-convective currents with respect to the case in which they are absent can have a stabilizing or destabilizing effect according to whether

$$R_L\lambda\,s\alpha_c/\alpha_T-R_{cL}\,(s\lambda+\alpha_T/\alpha_c)N\lambda\,Le \gtrless 0. \quad (35)$$

Thus, in the case of heating from below and salting from above with $s = 1$, the anisotropic effects have a stabilizing effect; while in the case of heating from above and salting from below the anisotropic effect is a destabilizing one. For oscillatory stability (i.e. $\omega \neq 0$) we write from (28)

$$F(\lambda) = a(\lambda)R+b(\lambda)R_c \quad (36)$$

where

$$a(\lambda) = \{Sc^2[Pr\,F+2\pi^2(1+\lambda^2 N\,Le)]-s2\pi^2\lambda\alpha\,s_c/\alpha_T\,Pr\,Sc\}/D(\lambda) \quad (37)$$

$$b(\lambda) = Pr^2\{2\pi^2\lambda N\alpha_T/\alpha_c-s(2\pi^2+Sc\,F)\}/D(\lambda) \quad (38)$$

with

$$D(\lambda) = [Pr+Sc(1+\lambda^2 N\,Le)]\times\{2\pi^2+F(Pr\,(2\pi^2+Sc\,F)+2\pi^2\,Sc(1+\lambda^2 N\,Le))\}. \quad (39)$$

Also from equations (34) we write

$$G(\lambda) = c(\lambda)R + d(\lambda)R_c \quad (40)$$

where

$$c(\lambda) = Sc/[Pr + Sc(1 + N\lambda^2 Le)] \quad (41)$$

$$d(\lambda) = -sPr/[Pr + Sc(1 + N\lambda^2 Le)]. \quad (42)$$

We consider the difference

$$e^2[F(\lambda) - F(0)] = f_1 R + f_2 R_c \quad (43)$$

where

$$e^2 = D(\lambda)D(0) \quad (44)$$

$$f_1 = \{Sc^2[Pr\,F + 2\pi^2(1 + \lambda^2 N\,Le)] - 2s\lambda\alpha_c/\alpha_T\,Pr\,Sc\,\pi^2\}D(0) - Sc^2(2\pi^2 + Pr\,F)D(\lambda) \quad (45)$$

$$f_2 = Pr^2\{2\lambda\pi^2 D(0)\alpha_T/\alpha_c + s(2\pi^2 + Sc\,F)[D(\lambda) - D(0)]\}. \quad (46)$$

The stabilizing or destabilizing effect of anisotropy depends on the sign of the difference $F(\lambda) - F(0)$ together with the condition max $[G(\lambda),\ G(0)] \leqslant 4\pi^2$. Thus from (28), (34) and (43) it follows that the anisotropic effects have a stabilizing or destabilizing effect according to

$$f_1 R + f_2 R_c \gtrless 0 \quad (47)$$

and

$$\max[G(\lambda), G(0)] \leqslant 4\pi^2. \quad (48)$$

As a particular case, from (45) to (47) it follows that, for $s = 1$, $\beta = 0$ and $\Gamma = 0$, the effect of the anisotropic currents is stabilizing, while for $\beta = 0$, and $\Gamma < 0$, the effect is destabilizing. Furthermore it seems worth mentioning that the same type of behaviour also occurs for convective instability in a binary mixture of two fluids without a porous medium (Maiellaro and Palese [7]).

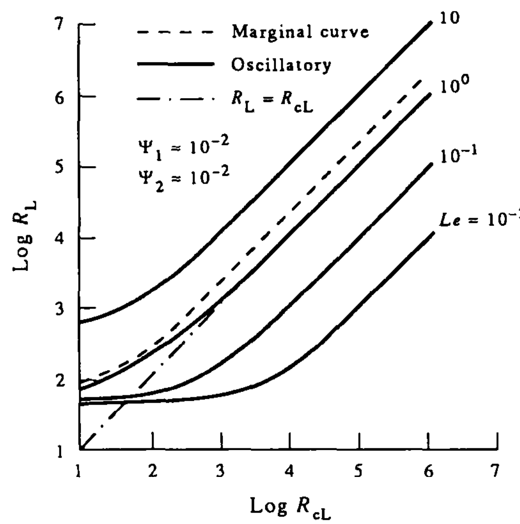

FIG. 2. Stability curves for different values of Le.

4. RESULTS

The linear stability of a binary mixture of two miscible fluids in a horizontal porous layer is investigated. The linear theory is based on the normal mode analysis under the assumption that the amplitudes of the convection are small. The effect of anisotropic thermo-convective currents in the presence of Soret and Dufour currents on the critical Rayleigh numbers for both marginal and overstable motions are determined.

The effect of Prandtl number on the stability is

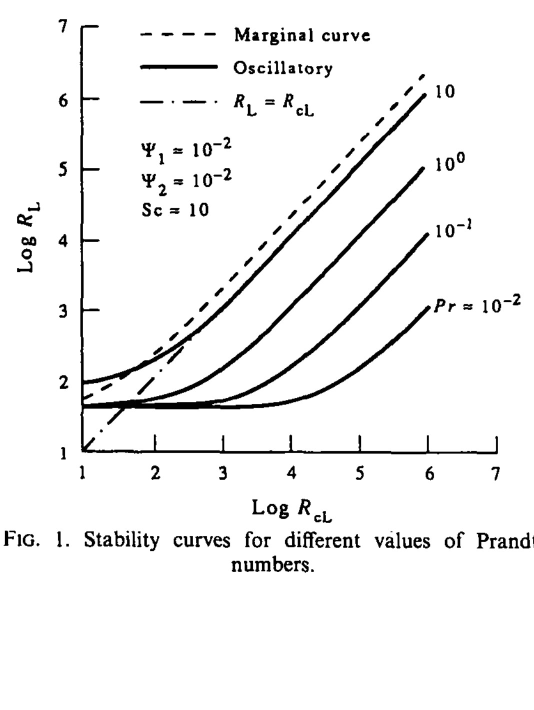

FIG. 1. Stability curves for different values of Prandtl numbers.

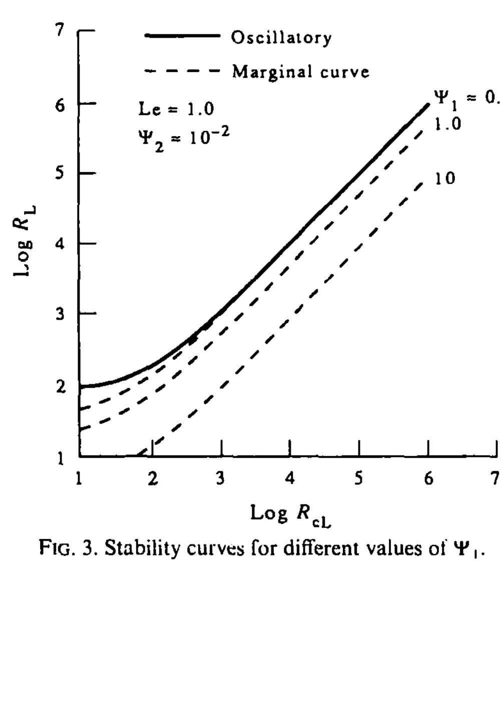

FIG. 3. Stability curves for different values of Ψ_1.

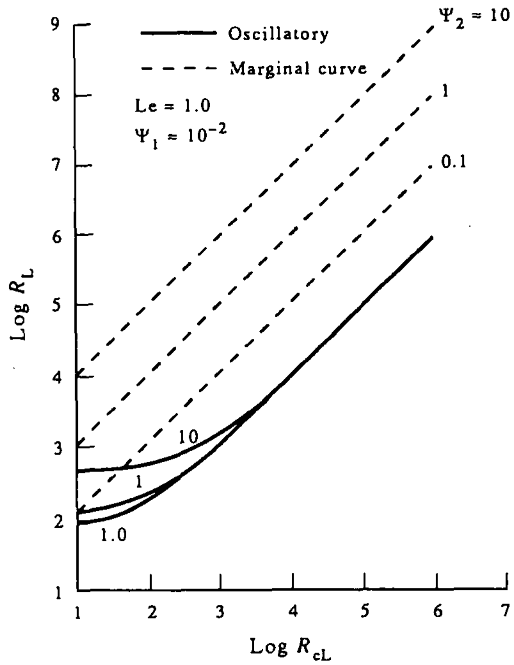

FIG. 4. Stability curves for different values of Ψ_2.

shown in Fig. 1. We observe from Fig. 1 that the effect of increasing Prandtl number is to stabilize the system. Its effect is more pronounced for large values of the concentration Rayleigh number. For small values of R_{cL} and large σ the overstable motions are ruled out, whereas for small *Pr* overstable motions are always possible.

The effect of Lewis number (the ratio of diffusivities) is depicted in Fig. 2. The effect of the ratio of diffusivities on the stability is found to be similar to that of Prandtl number.

The effect of separation parameter $\psi_1(=\lambda\alpha_c/\alpha_T)$ on the stability of the system is shown in Fig. 3. We observe from Fig. 3 that the increasing effect of ψ_1 is to destabilize the system.

The effect of the parameter $\psi_2(=\lambda^2 N)$ is shown in Fig. 4. It is evident from this figure that for the typical values considered, the marginal stability occurs as the first bifurcation.

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